NASA TECHNICAL MEMORANDUM 38

REPORT NO. 53914

THE GRAVITATIONAL RED SHIFT

By Peter Eby
Space Sciences Laboratory

September 16, 1969

NASA

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama

NASA TECHNICAL MEMORANDUM 38

REPORT NO. 53914

THE GRAVITATIONAL RED SHIFT

By Peter Eby
Space Sciences Laboratory

September 16, 1969

NASA

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama

INTERNAL NOTE

IN-SSL-N-68-19
Changed to TM X-53914
September 16, 1969

THE GRAVITATIONAL RED SHIFT

By

Peter Eby

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama 35812

ABSTRACT

The general relativistic red shift effect and its proposed measurement by means of the Hydrogen Maser Clock is discussed. Some aspects of the problem are considered which are not referred to in the standard texts on relativity. It is hoped that this discussion will be a more realistic and thorough consideration of the problem. In particular, the effect of the gravitational field on the physical mechanism of the atomic clock is discussed.

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

THE GRAVITATIONAL RED SHIFT

By

Peter Eby

SPACE SCIENCES LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

TABLE OF CONTENTS

INTRODUCTION

The general relativistic red shift effect and its proposed measurement by means of the Hydrogen Maser Clock is discussed. Some aspects of the problem are considered which are not referred to in the standard texts on relativity. It is hoped that this discussion will be a more realistic and thorough consideration of the problem. In particular, the effect of the gravitational field on the physical mechanism of the atomic clock is discussed.

GENERAL DISCUSSION

In the analysis of the Maser Clock Relativity Experiment, one has three physical systems to consider: (1) an atomic system (in this case, a hydrogen atom), (2) the electromagnetic field, and (3) the gravitational field. The effect to be measured (i.e., a frequency shift) will be a result of the interaction of these three systems.

In the experiment, hydrogen atoms in the ground state undergo a hyperfine transition producing a radiation field at a particular point in a gravitational field. That radiation field is propagated through the gravitational field to another point in space where its frequency is compared with that of radiation produced by other hydrogen atoms at that point. So one has two processes: (a) the emission of radiation due to a hyperfine transition, and (b) the propagation of an electromagnetic wave through a gravitational field. Process (a) involves the interaction of systems (1), (2), and (3), process (b) the interaction of systems (2) and (3).

This seems to be the most realistic way to look at the experiment. However, it is not the way the experiment is usually discussed in the standard texts. Usually the comment is made that the infinitesimal interval of proper time $d\tau$ is given by

$$d\tau = \sqrt{-g_{00}}\, dt \quad ,$$

where g_{00} is the time-time component of the metric tensor and dt the infinitesimal interval of coordinate time. This proper time is considered to be the time measured by an "ideal" or "standard" clock placed at the particular point

in question. The so-called "ideal" clock is considered as the measure of proper time in the same sense as a standard meter stick is a measure of spatial interval. Implicit in this is the assumption that all "ideal" clocks always behave the same way. Atomic clocks such as the Maser Clock are usually considered to be examples of such ideal clocks. Then the difference in frequency observed when two Maser Clocks at different points are compared is simply a result of the difference in $\sqrt{-g_{00}}$.

This is an oversimplified view of the actual situation, and is valid only when the following conditions are met:

I. In process (a), the effect of system (3) on systems (1) and (2) is completely negligible.

II. In process (b), the frequency expressed in units of coordinate time is constant as the wave propagates from one point to another in space.

Another way of stating condition I is that the gravitational field must not have any effect on the actual mechanism of the clock under consideration. An example of a clock for which this condition would not be met is a pendulum clock. The frequency of this clock is proportional to $\sqrt{g}$, where g is the acceleration due to gravity at the particular point where the clock is placed. If one pendulum clock is placed on the moon and another on the earth and their frequencies compared, the ratio of the earth clock's frequency to the moon clock's frequency is $\sqrt{6}$. This large "red shift" would have nothing to do with general relativity or any time dilation effect. One can also see that in the limit of very high gravitational fields, the performance of just about any imaginable clock would be altered to the point of its being crushed.

This matter has been expressed rather clearly by V. A. Fock [1]:

"In general, however, no theory is capable of predicting how a clock will behave when subjected to impacts or arbitrary accelerations (locally equivalent to gravitational fields) without entering into the details of the clock construction.

"One can also introduce the notion of clocks nearly or, in the limit, entirely insensitive to impacts and accelerations. One could then propose to measure proper time by the reading of such ideally insensitive clocks, this being the physical meaning of proper time. But one should note that such a

proposal, although not in contradiction to the theory of relativity, does not follow from it and represents a special hypothesis."

So one must look at process (a) in more detail to see whether the condition I is met for the particular case of the Maser Clock. It must be remembered that the frequency shifts to be measured are very small so that any effects such as the above effect on pendulum clocks must be negligible compared to the measured small frequency shifts.

ATOMIC PROCESSES

The first effect that occurs to one is a gravitational Stark effect. This is analogous to the usual Stark effect in which the energy levels of an atom are shifted in the presence of a uniform static electric field. The gravitational field could conceivably affect the atomic energy levels in a similar way.

The Hamiltonian for a hydrogen atom with the proton fixed in a uniform gravitational field would be

$$H = H_o + mgz \quad ,$$

where H_o is the usual unperturbed Hamiltonian, m the electron mass, and g the acceleration due to gravity. Treating mgz as a perturbation, one finds [2]* that the ground state is unaffected, since it has even parity, but the first excited state has two components which are shifted in energy by an amount

$$\Delta E = \pm 3a_o mg \quad ,$$

where a_o is the Bohr radius. This shift turns out to be approximately 10^{-15} that of the hyperfine structure splitting which indeed seems small. But for a hydrogen maser one can observe shifts of 10^{-14} that of the hyperfine structure. So, if the above shift were to occur in the ground state it would be near the threshold of observability. This shift does not occur in the ground state, but the discussion illustrates the caution one must use in arguing that these effects are "small" and thus can be neglected.

---

*Note that eE is replaced by mg.

What actually happens when an atomic system is placed in a gravitational field is that the electromagnetic field is altered. To see how this occurs, look at Maxwell's equations for the field quantities F_{ik} in an arbitrary Riemannian metric given by g_{ik}. They are [3]:

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$$

$$\frac{1}{\sqrt{-g'}} \frac{\partial}{\partial x^k} \left[\sqrt{-g'}\, F^{ik}\right] = -\frac{4\pi}{c} j^i \qquad (i, k, l) = 0, 1, 2, 3,$$

where g' is the determinant of g_{ik}, j^i is the current four vector, and repeated indices are summed. The presence of the expressions involving the metric ($\sqrt{-g'}$, $F^{ik} = g^{il} g^{km} F_{lm}$) describes the effect of the gravitational field on the electromagnetic field quantities. Making the definitions

$$\vec{E} = (F_{10}, F_{20}, F_{30}) \qquad \vec{j} = \sqrt{-g'}(j^1, j^2, j^3)$$

$$\vec{B} = (F_{23}, -F_{13}, F_{12}) \qquad c\rho = \sqrt{-g'}\, j^0$$

$$\vec{D} = \sqrt{-g'}(F^{01}, F^{02}, F^{03})$$

$$H = \sqrt{-g'}(F^{23}, -F^{13}, F^{12}) \ .$$

One can write the above equations as

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j} \qquad \vec{\nabla} \cdot \vec{D} = 4\pi\rho$$

which are identical to the ordinary Maxwell equations in a dielectric medium. The gravitational field behaves like a dielectric and refracts the light. It also has an effect on electrostatics. To see this, look at the case where the metric is of the form

$$g_{11} = g_{22} = g_{33} = a$$

$$-g_{00} = b \qquad g_{ik} = 0 \quad \text{for } i \neq k \quad .$$

The isotropic form of the Schwarzschild metric is of this type. Then

$$\vec{D} = \sqrt{-g}\, g^{00} \left(g^{11} F_{01},\ g^{22} F_{02},\ g^{33} F_{03} \right)$$

$$= \sqrt{a^3 b} \left(-\frac{1}{b} \right) \left(-\frac{F_{10}}{a},\ -\frac{F_{20}}{a},\ -\frac{F_{30}}{a} \right)$$

$$= \sqrt{a/b}\ \vec{E} = \epsilon \vec{E} \quad \text{where } \epsilon = \sqrt{a/b} \quad ,$$

and similarly,

$$\vec{H} = \frac{1}{\mu} \vec{B} \quad \text{where } \mu = \sqrt{a/b} \quad .$$

For a point charge e at rest in the above metric, one has as a solution of Maxwell's equations

$$\phi = \frac{e}{\epsilon r} \equiv \frac{1}{\epsilon} \phi_o \qquad \vec{E} = \vec{\nabla} \phi \quad ,$$

if one assumes g_{ik} is constant over the region where the field is being evaluated. Similarly, the magnetic field of a magnetic dipole is given by

$$\vec{H} = \frac{1}{\mu} \left[\frac{3\vec{n}\,(\vec{n} \cdot \vec{m}_1) - \vec{m}_1}{r^3} \right] \equiv \frac{1}{\mu} \vec{B}_o$$

where $\vec{m}_1$ is the magnetic dipole moment and $\vec{n}$ is a unit vector in the direction of the radius vector. In the case where

$$a = \left(1 + \frac{2GM}{c^2R}\right) ; b = \left(1 - \frac{2GM}{c^2R}\right)$$

(i.e., the isotropic Schwarzschild metric for large R, where M is the mass of the attracting body (the earth), G is the gravitational constant, and R is radial distance to the point where the atom is located), one sees that

$$\phi = \left(1 - \frac{2GM}{c^2R}\right)\phi_o \quad \vec{H} = \left(1 - \frac{2GM}{c^2R}\right)\vec{B}_o$$

to first order in $\frac{2GM}{c^2R}$. So, if we associate $-e\phi$ and $\vec{m}_2 \cdot \vec{H}$ with the potential energy of an electron with magnetic moment $\vec{m}_2$ in the field $(\vec{E}, \vec{H})$ of a fixed proton with charge $+e$ and magnetic moment $\vec{m}_1$, one can see that the potential energy is shifted by a factor $\left(1 - \frac{2GM}{c^2R}\right)$. From this, one would expect that the same sort of shift would be present in the correct quantum mechanical treatment of the hydrogen atom in the presence of a gravitational field. This is another way of saying that condition I does not appear to be satisfied.

This argument is confirmed by calculations based on the Dirac equation in curved space. Oliveira and Tiomno [4] have shown that the Dirac equation predicts shifted energy levels for a hydrogen atom in a gravitational field, and a similar calculation for the Birkhoff theory [5] (a linearized gravitational theory) gives the same result. The levels are supposedly shifted absolutely by an amount precisely equal to the expected red shift due to the g_{00} component of the metric tensor.

It is important to realize that the crucial assumption here is that the nucleus of the atom is fixed in the gravitational field. For example, this situation could be realized for an atom fixed in a crystal lattice. (Note that this is the situation for the red shift measurement using the Mossbauer effect, although the transitions in question there are nuclear rather than atomic. We do not know how to treat the case of interacting nuclear and gravitational fields so there is no way of telling whether a similar effect would be predicted.)

If the atoms are in free fall, rather than fixed, no such effect as the one described above occurs since the metric can be transformed to the Minkowskian form in the rest frame of the atom in question. It is also clear that the effects of field gradients on the energy levels are completely negligible.

The consideration of the Stark effect shows that when the field departs from 0 to 1 g the effects are negligible. It is also easy to convince oneself that allowing g_{ik} to vary over the dimensions of the atoms will have negligible effect on the previous results involving electrostatics.

So, if the preceding calculations are correct, it appears that the frequency shift expected from a maser placed in a gravitational field will depend on the extent to which the emitting atoms are in free fall. They certainly are in free fall part of the time, but not all of the time. And what the average frequency will be, including the extra Doppler effects associated with the free fall, appears difficult to predict.

This completes the analysis of process (a). The results are inconclusive, but we point out that the elliptic orbit proposal will allow us to eliminate the above uncertainties. The clock in orbit is in a locally Galilean frame at all times. So any effects such as those described above would be present only in the ground maser, and the readings of the orbital clock at different altitudes would thus yield the red shift independent of any constant effects in the ground maser. In addition, it is suggested that the issue under discussion here could be further tested by some sort of centrifuge experiment with a maser subject to 1-g radial force as well as a vertical 1-g force.

ELECTROMAGNETIC WAVES

The analysis of process (b) is carried out by looking for wave-like solutions to the Maxwell equations given previously with the effects of the metric g_{ik} included. The Schwarzschild metric is of interest here, but let us for simplicity look first at the metric

$$g_{11} = g_{22} = g_{33} = 1 \ , \ g_{00} = -\left(1 + \frac{gx}{c^2}\right)^2 ,$$

where g is a constant. This metric characterizes a uniform gravitational field and so it approximates the Schwarzschild metric over a sufficiently small area. We also note that this metric is an exact solution to the field equations for matter-free space. The Christoffel symbols are

$$\Gamma^1_{00} = -\frac{g}{c^2}\left[1 + \frac{gx}{c^2}\right], \quad \Gamma^0_{10} = \frac{g}{c^2}\left[1 + \frac{gx}{c^2}\right]^{-1} .$$

Writing the Maxwell equations in the form

$$\frac{D}{Dx^k} F_i^{\ k} = 0, \quad \frac{D}{Dx^l} F_{ik} + \frac{D}{Dx^i} F_{kl} + \frac{D}{Dx^k} F_{il} = 0,$$

where D/Dx^k denotes the covariant derivative with respect to x^k, one can combine the equations to get

$$0 = g^{ab} \frac{D}{Dx^a} \left[\frac{D}{Dx^b} F_{ik} + \frac{D}{Dx^i} F_{kb} + \frac{D}{Dx^k} F_{bi} \right]$$

$$= g^{ab} \frac{D}{Dx^a} \frac{D}{Dx^b} F_{ik} + \frac{D}{Dx^i} \left[\frac{D}{Dx^a} F^a_{\ k} \right]$$

$$- \frac{D}{Dx^k} \left[\frac{D}{Dx^a} F^a_{\ i} \right] ,$$

where one has used the fact that the covariant derivatives commute, since the curvature tensor is zero for this metric, and that the covariant derivative of g_{ik} is zero. The two bracketed quantities in the last equation are zero, thus

$$g^{ab} \frac{D}{Dx^a} \frac{D}{Dx^b} F_{ik} = 0 \ .$$

This is a sort of generalized D'Alembertian. Employing the usual expressions for the covariant derivative and the above Christoffel symbols, one gets, to first order in g/c^2, after a lengthly calculation

$$0 = \Box F_{12} + \frac{3g}{c^2} \frac{\partial F_{12}}{\partial x} , \quad 0 = \Box G_{03} + \frac{3g}{c^2} \frac{\partial G_{03}}{\partial x} ,$$

$$0 = \Box F_{23} + \frac{g}{c^2} \frac{\partial F_{23}}{\partial x} , \quad 0 = \Box G_{01} + \frac{g}{c^2} \frac{\partial G_{01}}{\partial x} ,$$

$$0 = \Box F_{13} + \frac{3g}{c^2} \frac{\partial F_{13}}{\partial x} + \frac{2g}{c^2} \frac{\partial F_{23}}{\partial y} , \quad 0 = \Box G_{02} + \frac{3g}{c^2} \frac{\partial G_{02}}{\partial x} - \frac{2g}{c^2} \frac{\partial G_{01}}{\partial y} ,$$

where

$$G_{o\alpha} = F_{o\alpha} \Big/ \left[1 + \frac{gx}{c^2}\right], \quad \Box F_{ik} = g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} F_{ik} \quad ,$$

and we have assumed F_{ik} to depend only on $x = x^1$ and $y = x^2$. It is interesting to note that the terms involving first derivatives in F_{ik} indicate that the electromagnetic wave is attenuated.

Two explicit solutions to these equations will be given, valid to first order in g/c^2. The first describes a wave traveling in the x direction and has only x and t dependence.

$$F_{12} = -G_{02} = C \exp \left\{ -\frac{g}{c^2} x + i\left[k\left(1 - \frac{gx}{2c^2}\right) x - \omega t\right]\right\},$$

where C, k, and ω are constants and t indicates coordinate time ($ct = x^\circ$). This solution satisfies the first-order Maxwell equations, as can be checked. Also $k = \omega/c$. The wave is transverse, and it is attentuated as expected. It is no longer a plane wave and the wave number is modulated by a factor $\left(1 - \frac{gx}{2c^2}\right)$. The solution can be written as

$$F_{12} = -G_{02} = A(x)\, e^{-i\omega t} \quad ,$$

and, if one is located at a particular point in space, one has

$$\omega t = \omega \frac{\tau}{\sqrt{-g_{00}}} = \frac{\omega}{\sqrt{-g_{00}}} \tau \quad .$$

So the frequency in terms of proper time exhibits the usual red shift formula.

There also exists a solution of the form

$$F_{12} = C \exp i\left[k\left(1 - \frac{gx}{c^2}\right) y - \omega t\right] \quad ,$$

valid to first order in g/c^2 with a corresponding solution for G_{01} and G_{02} with the same phase dependence. This solution represents a wave traveling in the y direction and being bent slightly in the x direction, exhibiting the well-known gravitational deflection. There appears to be no change in the polarization here. Solutions to all orders in g/c^2 surely exist with the same general properties as those given above, the frequency shift appearing in the same way. It is assumed in all this that the electromagnetic wave has negligible effect on the background metrics.

The solution for the Schwarzschild metric can presumably be found in the same way as in the above example although the computational difficulties are greatly increased. It would be interesting to see if wavelike solutions to the Maxwell equations exist which represent plane waves impinging on the gravitational field of a spherical mass and whether these solutions exhibit the usual bending. It is clear that the red shift will come out in the same way as in the previous example since the solutions can be written as

$$F_{ik} = C_{ik}(x, y, z) \exp - i\omega t \quad ,$$

where the C_{ik}'s are functions of x, y, and z. It also seems reasonable to expect that the attenuation noted previously will be present. This attenuation has not been observed in the literature. This effect could possibly alter the energy content estimates for Quasars, since these estimates are based on apparent luminosity. The large red shift associated with these objects could be accompanied by an attenuation if the red shift is gravitational, thus making the problem of accounting for the power output even more difficult. In addition, for the Schwarzschild field, since the curvature tensor is nonzero, one could get polarization effects which could be checked experimentally. However, in the absence of an explicit solution, this is only speculation. The only result which seems clear at present is that one gets the usual red shift for the Schwarzschild field; that is, condition II appears to be valid.

To treat the problem of interacting gravitational and electromagnetic fields quantum mechanically appears to be a formidable problem, to say the least. Even the problem of a classical gravitational field interacting with the quantized electromagnetic field appears extremely difficult, despite the derivations in many texts involving "photons." It seems, however, that treating both fields classically is quite adequate for the present situation, and one sees how this has given the expected red shift.

CONCLUSION

First, it has been shown that the effect of the gravitation field on the mechanism of clocks cannot in general be disregarded, although the elliptical orbit proposal offers a way to eliminate effectively any such effect for the Maser Clock Experiment. Second, the solutions to Maxwell's equations in curved space give the usual expression for the red shift.

REFERENCES

1. Fock, V. A.: Theory of Space, Time, and Gravitation. MacMillan Co., New York, pp. 43, 235, 1964.

2. Merzbacher, E.: Quantum Mechanics. John Wiley and Sons, New York, p. 389, 1963.

3. Landau, L. D., and Lifshitz, E. M.: Classical Theory of Fields. Addison Wesley, p. 288, 1962.

4. Oliveira, C. G., and Tiomno, J.: Nuovo Cimento. 24, 672, 1962.

5. Moshinsky, M.: Phys. Rev. 80, 514, 1950.